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On the Convergence of Certain Classes of Series of Functions.

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§ 1. INTRODUCTION.

Let $v_n(x)$, $n = 0, 1, 2, \cdots$, be an infinite sequence of functions of x which may be written in the form

$$v_n(x) = \sum_{i=0}^{\mu_1} \sum_{j=0}^{\nu_1} a^{(n)}_{\mu-i,\,\nu-j} \; x^{\mu-i}(\log x)^{\nu-j} + \frac{M_n(x)}{x^{\mu_1-\mu+1}(\log x)^{\nu_1-\nu+1}} \tag{1}$$

where μ, μ_1, ν, ν_1 are integers such that $\mu_1 \geqq \mu$, $\nu_1 \geqq \nu$, and where $M_n(x)$, for fixed n, is a function of x which in absolute value is not greater than a constant M_n when $|x|$ is greater than some constant X and x is in a sector V formed by two rays proceeding from zero to infinity and including the positive axis of reals between them. Let $M_n(x)$, for every n, be an analytic function of x in every finite portion of the region of the x-plane just defined.

Moreover, let one of the coefficients a in (1), say $a^{(n)}_{kl}$ where k and l are not simultaneously zero, possess the following properties:

1) As n becomes infinite $a^{(n)}_{kl}$ becomes infinite, while its argument approaches a finite limit, say that it comes to coincide with the argument of a given constant σ, so that it may be written $a^{(n)}_{kl} = \sigma(\alpha^{(n)} + i\beta^{(n)})$ where $\alpha^{(n)}$ and $\beta^{(n)}$ are real. Suppose, moreover, that $\alpha^{(n)}$ is monotonic increasing when n is greater than some given constant. (It is easy to see that $\beta^{(n)}/\alpha^{(n)}$ approaches zero as n becomes infinite, a fact for which we shall have use later.)

2) The coefficient $a^{(n)}_{kl}$ has a dominance property of such sort that

$$\lim_{n=\infty} M_n(x)/a^{(n)}_{kl} = 0$$

for every x for which $M_n(x)$ is analytic and

$$\lim_{n=\infty} a^{(n)}_{ij}/a^{(n)}_{kl} = 0$$

unless simultaneously $i = k$ and $j = l$.

3) When $v_n(x)$ is one-termed so that it has the special value $v_n(x) = a^{(n)}_{kl} x^k (\log x)^l$ and $a^{(n)}_{kl}$ is real we make no further hypothesis; otherwise we

suppose that a positive constant ϵ_1 exists such that when n is greater than some appropriate N_{ϵ_1} we have $a^{(n+1)} - a^{(n)} \geqq \epsilon_1$.

Then let us consider the series of functions

$$S(x) = \sum_{n=0}^{\infty} c_n e^{v_n(x)} \tag{2}$$

where $c_0, c_1, c_2 \cdots$ are constants.

A point x will be called an exceptional or a non-exceptional point for the series $S(x)$ according as x is or is not a singularity of $v_n(x)$ for some value of n.

The principal object of the present paper is to consider the central convergence problem for the series $S(x)$ and for a certain other series defined in § 4 and having similar properties. It turns out that the convergence theory of these series may be readily developed in an elementary way. The character of the region of convergence and the uniform convergence of series $S(x)$ are treated in § 2. In § 3 theorems are established showing the coincidence in special cases of the regions of convergence of series $S(x)$ depending on different sequences $v_n(x)$. In § 4 are developed corresponding properties of a second class of series there defined. In § 5 cases are considered in which series of either class define functions having formal power series expansions.

A considerable variety of important classes of series are included in the general form (2). We shall exhibit a few of these.

(a) Let $v_n(x) = n \log x$. Our series is then an ascending power series. If we take $v_n(x) = -n \log x$ we have the descending power series.

(b) If we put $v_n(x) = -\lambda_n x$ where $\lambda_0, \lambda_1, \lambda_2, \cdots$ is an increasing sequence of real numbers tending to infinity then $S(x)$ is the generalized Dirichlet series. The extension of Dirichlet series employed in *Transactions American Mathematical Society* 17 (1916), p. 218, is also essentially included here.

(c) If we take $v_n(x) = \log \Gamma(x) - \log \Gamma(x+n)$, so that our series becomes the factorial series, it may be shown that $v_n(x)$ fulfills the requisite conditions, the asymptotic formula for $\Gamma(x)$ serving readily for this purpose.

(d) Indeed the more general class of series $\Sigma\, c_n g(x+n)$ which I have treated in *Trans. Amer. Math. Soc.* 17 (1916), pp. 207-232, may be shown to belong to the class treated here.

In the convergence proofs we shall have need of two lemmas which are reproduced here for the reader's convenience (for reference to the proofs of these lemmas see page 211 of the paper just cited):

Lemma I. *Let* $a_0 + a_1 + a_2 + \cdots$ *be a convergent series of constants and let* $\beta_0, \beta_1, \beta_2, \cdots$ *be an infinite sequence of numbers such that the series*

$\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ *is convergent. Then the series* $\alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots$ *is convergent.*

Lemma II. *Let* $\alpha_0 + \alpha_1 + \alpha_2 + \cdots$ *be a convergent series of constants and let* $\beta_0, \beta_1, \beta_2, \cdots$ *be an infinite sequence of functions of the complex variable* x *analytic in a given closed domain* D *and such that the series* $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ *converges uniformly in* D. *Then the series* $\alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots$ *converges uniformly in* D.

§ 2. *Character of the region of convergence of* $S(x)$.

Let us suppose that the series $S(x)$ converges for a given non-exceptional value x_0 of x, say briefly that $S(x_0)$ converges; and let us seek conditions on the non-exceptional value x_1 of x sufficient to ensure that $S(x_1)$ shall be convergent. We employ lemma I, taking

$$\alpha_n = c_n e^{v_n(x_0)}, \quad \beta_n = e^{v_n(x_1) - v_n(x_0)}.$$

The series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$, whose convergence is sufficient to ensure the convergence of $S(x_1)$, may be put in the form

$$\sum_{n=N}^{\infty} \left| e^{\alpha^{(n+1)}\tau} - e^{\alpha^{(n)}\tau} \right| r_n \tag{3}$$

where

$$r_n = \frac{\left| e^{v_{n+1}(x_1) - v_{n+1}(x_0)} - e^{v_n(x_1) - v_n(x_0)} \right|}{\left| e^{\alpha^{(n+1)}\tau} - e^{\alpha^{(n)}\tau} \right|},$$

τ having the value

$$\tau = \theta\,\sigma\,\{x_1^k(\log x_1)^l - x_0^k(\log x_0)^l\},$$

where θ is a positive constant not greater than unity. We shall now suppose that the real part $R(\tau)$ of τ is negative.

We shall now show that a proper choice of θ will bring it about that r_n is bounded. If $v_n(x)$ is one-termed in the sense of condition 3) then we take $\theta = 1$, whence $r_n = 1$; otherwise we take θ to be less than unity. Then if we divide the numerator and the denominator of the fraction r_n by $| e^{\alpha^{(n)}\tau} |$ we have left in the denominator a quantity bounded away from zero for every n in (3) is N is taken sufficiently large, as one sees from the condition in hypothesis 3). Moreover, each of the two exponential terms in the numerator of this fraction, and hence this numerator itself, approaches zero as n becomes infinite, as one sees readily through use of 1), 2) and 3) and particularly of the dominance property of $a^{(n)}_{kl}$. Hence in any case r_n is bounded.*

It follows therefore that series (3) converges provided that the series

$$\sum_{n=N}^{\infty} \left| e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} \right|$$

is convergent.

To prove the convergence of this series we observe that

$$e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} = (1/\tau)\int_{a^{(n)}}^{a^{(n+1)}} e^{u\tau}du$$

so that

$$\left| e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} \right| \leqq (1/|\tau|)\int_{a^{(n)}}^{a^{(n+1)}} e^{uR(\tau)}du.$$

The series of which the nth term is the second member of this relation is obviously convergent since $a^{(n)}$ is ultimately monotonic and $R(\tau) < 0$ on account of the condition imposed on x_1.

Hence we have the part of the following theorem which refers to convergence (not absolute convergence):

Theorem I. *Let x_0 and x_1 be two values of x which are non-exceptional for the series $S(x)$ and suppose that $S(x_0)$ converges [converges absolutely]. Then $S(x_1)$ also converges [converges absolutely] provided that*

$$R\{\sigma x_1^k(\log x_1)^l\} < R\{\sigma x_0^k(\log x_0)^l\}.$$

The proof of the part of the theorem referring to absolute convergence is immediate. In fact it is sufficient to show that the ratio ${}_ev_n(x_1)/{}_ev_n(x_0)$ is bounded as n becomes infinite; and this is an immediate consequence of the hypotheses on $v_n(x)$.

By a *region C of convergence* of the series $S(x)$ we shall mean a region such that $S(x)$ converges for every non-exceptional value of x in the interior of C and diverges for every non-exceptional value of x exterior to C. In a similar way we define a *region Γ of absolute convergence of $S(x)$.*

By means of theorem I and the application of a classic method it is easy to determine the character of the regions of convergence and absolute convergence of $S(x)$. Compare the similar argument on p. 214 of the memoir already cited. We have the following result:

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* It should be observed that the only use of condition 3) in this proof is that made in showing that r_n is bounded, so that the theorem obtained is true when r_n is bounded even though condition 3) should not be satisfied.

Theorem II. *There exists a unique real number* λ $[\mu]$ *such that the region of convergence [absolute convergence] of the series* $S(x)$ *is bounded by the curve*

$$R\{\sigma x^k (\log x)^l\} = \lambda \quad [= \mu] \tag{4}$$

and lies on that side of this curve for which $R\{\sigma x^k (\log x)^l\}$ *is less than* λ $[\mu]$.

By the use of lemma II and a modification, mostly verbal in character, of the argument by which theorem I was established we may prove the following theorem:

Theorem III. *The series* $S(x)$ *converges uniformly in any closed domain* D *which lies within its region of convergence and contains no point which is exceptional for* $S(x)$ *or is a limit point of points which are exceptional for* $S(x)$.

As an immediate consequence of this we have the following:

Theorem IV. *The sum of the series* $S(x)$ *is a function* $S(x)$ *of* x *which is analytic at every non-exceptional point which is in the interior of its region of convergence and is not a limit point of exceptional points; and the derivatives of* $S(x)$ *at every such point may be found by differentiating the series* $S(x)$ *term by term.*

We should examine briefly the nature of the curves defined by equations of the form (4). For the case when $l = 0$ and k is positive I have already briefly described them in *Bull. Amer. Math. Soc.* (2) 23 (1917), pp. 424-425. In this case they are obviously algebraic. In particular, when $l = 0$ and $k = 1$ the curve is a straight line.

Suppose next that $l = 0$ and k is a negative integer $-t$. Then the curves have equations of the form $R(\sigma x^{-t}) = \eta_1$ and are again algebraic. If we write $\sigma = |\sigma| e^{i\phi}$, $\eta = \eta_1/|\sigma|$ and $x = re^{i\theta}$ where r is real and not negative, the equation of our curve may be written in polar coördinates r and θ in the form

$$\eta r^t = \cos(\phi - t\,\theta).$$

In case $\eta = 0$ our curve consists of $2t$ rays proceeding from 0 to ∞ and dividing the angular space about zero into $2t$ equal parts or sectors. The quantity $\cos(\phi - t\,\theta)$ is negative within alternate sectors of this set (the sectors of convergence) and positive within the others (in general the sectors of divergence). When η is not zero the curve consists of t branches lying within alternate sectors of the preceding set of $2t$ sectors, and in those for which $\cos(\phi - t\,\theta)$ is negative or positive according as η is negative or positive. For the case when $t = 1$ this curve is a circle.

When $k = 0$, $l = 1$ and σ is real equation (4) may be written in the form $R[\log x] = \eta_2$ or $|x| = \eta$, so that the curve is a circle about 0 as a center. As we have already seen, this includes the case when $S(x)$ is a power series either ascending or descending.

Except when $l = 1$ and $k = 0$, or $l = 0$ and k is unrestricted, the curves (4) are in general transcendental.

§ 3. *Coincidence of the regions of convergence of different series $S(x)$.*

In connection with series $S(x)$ let us consider the two related series

$$S_{kl}(x) = \sum_{n=0}^{\infty} c_n e^{a_{kl}^{(n)} x^k (\log x)^l}, \qquad \bar{S}_{kl}(x) = \sum_{n=0}^{\infty} c_n e^{\sigma a^{(n)} x^k (\log x)^l}.$$

It is clear that each of these is a series of the general class $S(x)$ defined in § 1. Consequently the four theorems already established are valid for these series also. In the present section we establish relations among them and the more general series $S(x)$.

Theorem V. *The boundary curve of the region of convergence [absolute convergence] is the same for the three series $S(x)$, $S_{kl}(x)$, $\bar{S}_{kl}(x)$.*

It should be observed that the theorem says nothing about convergence [absolute convergence] on the boundary of the region of convergence [absolute convergence].

It is obviously sufficient to prove the theorem for the case of any two pairs of the three series, say for $S(x)$ and $\bar{S}_{kl}(x)$, and for $S_{kl}(x)$ and $\bar{S}_{kl}(x)$. In fact, it is enough to prove it for the first of these pairs. The method of proof is identical in the two cases, so that it is sufficient to carry out the work for either one of them alone. For the latter pair the formulae are somewhat simpler than for the former, so that we shall give in detail the proof of only that part of the theorem which relates to the series $S_{kl}(x)$ and $\bar{S}_{kl}(x)$. This proof falls into two parts.

1. Let $\bar{x}$ be any non-exceptional point in the interior of the region of convergence of $S_{kl}(x)$. We shall prove that $\bar{x}$ is likewise in the interior of the region of convergence of $\bar{S}_{kl}(x)$.

If $R\{\sigma x^k (\log x)^l\} = \lambda$ is the boundary of the region of convergence of $S_{kl}(x)$, then $R\{\sigma \bar{x}^k (\log \bar{x})^l\} < \lambda$. Then it is obvious that non-exceptional numbers x_0 and x_1 exist such that

$$R\{\sigma \bar{x}^k (\log \bar{x})^l\} < R\{\sigma x_1^k (\log x_1)^l\} < R\{\sigma x_0^k (\log x_0)^l\} < \lambda. \tag{5}$$

Then x_0 and x_1 are points of convergence of $S_{kl}(x)$. Taking

$$\alpha_n = c_n e^{a_{kl}^{(n)} x_0{}^k (\log x_0)^l}, \qquad \beta_n = e^{\sigma \alpha^{(n)} x_1{}^k (\log x_1)^l - a_{kl}^{(n)} x_0{}^k (\log x_0)^l};$$

and applying lemma I we see that $\bar{S}_{kl}(x_1)$ converges if $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ converges. Now the exponent in the value of β_n may be written in the form

$$\alpha^{(n)} \{\sigma x_1{}^k (\log x_1)^l - \sigma x_0{}^k (\log x_0)^l\} - \beta^{(n)} i \sigma x_0{}^k (\log x_0)^l.$$

We can now proceed by the method employed for a like matter near the beginning of § 2 and show that $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ converges. Hence $\bar{S}_{kl}(x_1)$ converges; and therefore $\bar{S}_{kl}(\bar{x})$ converges, as one sees through theorem I and the relation between x and x_1 in (5); and $\bar{x}$ is in fact in the interior of the region of convergence of $\bar{S}_{kl}(x)$.

2. If we suppose next that $\bar{x}$ is in the interior of the region of convergence of $\bar{S}_{kl}(x)$ it may be shown that it is likewise in the interior of the region of convergence of $S_{kl}(x)$. For this it is sufficient to apply lemma I as in the preceding case, taking this time for α_n and β_n the values

$$\alpha_n = c_n e^{\sigma \alpha^{(n)} x_0{}^k (\log x_0)^l}, \qquad \beta_n = e^{a_{kl}^{(n)} x_1{}^k (\log x_1)^l - \sigma \alpha^{(n)} x_0{}^k (\log x_0)^l},$$

where x_0 and x_1 again satisfy relations (5), the curve $R\{\sigma x^k (\log x)^l\} = \lambda$ being now the boundary of the region of convergence of $\bar{S}_{kl}(x)$.

From the conclusions of the two preceding paragraphs we see that theorem V is true in so far as it relates to the region of convergence of $S_{kl}(x)$ and $\bar{S}_{kl}(x)$. The part relating to the region of absolute convergence of the same two series may be proved in the same way; or it may be proved more directly and more easily by a term-by-term comparison of the absolutely convergent series $S_{kl}(x_0)$ and $\bar{S}_{kl}(x_1)$ in the first case and $S_{kl}(x_1)$ and $\bar{S}_{kl}(x_0)$ in the second case, x_0 and x_1 being connected with an interior point $\bar{x}$ of the region of absolute convergence by a relation of the form (5), the curve $R\{\sigma x^k (\log x)^l\} = \lambda$ being now the boundary of the region of absolute convergence.

Theorem V thus established brings out the fact that for a given set of coefficients $c_0, c_1, c_2, \cdots$ in $S(x)$ the functions $v_n(x)$, subject to the permanent hypotheses as to character, can be modified in any way whatever so long as $a_{kl}^{(n)}$ is left unchanged (and indeed so long as σ and $\alpha^{(n)}$ are left unchanged) without disturbing the position of the boundary curve of the region of convergence. Such changes may introduce or remove exceptional points or modify their character; but beyond this it affects convergence [absolute convergence] only on the boundary of the region.

Again, *Theorem V affords us a satisfying means of finding the convergence number* λ [*the absolute convergence number* μ] *of the series* $S(x)$. In fact, it is the negative of the convergence abscissa [absolute convergence abscissa] of the Dirichlet series

$$D(t)=\sum_{n=0}^{\infty} c_n e^{-a^{(n)}t}$$

as one sees by comparison with $\bar{S}_{kl}(x)$ where $-t$ is thought of as replacing $\sigma x^k(\log x)^l$. Convenient formulae for the convergence abscissa of Dirichlet series are quoted or otherwise referred to on pages 224-225 of my memoir already cited.

§ 4. *Similar properties of a second class of series.*

Let us now consider similar questions for a series of the form

$$T(x)=c_0+\sum_{n=1}^{\infty} c_n P_1(x)P_2(x)\cdot\cdot\cdot P_n(x),$$

where $c_0, c_1, c_2, \cdot\cdot\cdot$ are constants and $P_1(x), P_2(x), P_3(x), \cdot\cdot\cdot$ a given sequence of functions.

A point $\bar{x}$ will be called exceptional for the series $T(x)$ if any one of the functions $P_n(x)$ has a singularity or a zero at $\bar{x}$; otherwise it will be called non-exceptional.

Let x_0 be a non-exceptional point such that $T(x_0)$ converges and let x_1 be a second non-exceptional point. Consider what relation between x_1 and x_0 is sufficient to ensure the convergence of the series $T(x_1)$. We employ lemma I, taking

$$a_n=c_nP_1(x_0)P_2(x_0)\cdot\cdot\cdot P_n(x_0),\quad \beta_n=\frac{P_1(x_1)\cdot\cdot\cdot P_1(x_1)}{P_1(x_0)\cdot\cdot\cdot P_n(x_0)}.$$

Then $T(x_1)$ converges if the series $\sum_n |\beta_{n+1}-\beta_n|$ converges. Now the ratio $R_n(x_1, x_0)$ of two consecutive terms of this series (the nth to the $(n-1)$th) may be put in the form

$$R_n(x_1,x_0)=\left|\frac{P_n(x_1)}{P_n(x_0)}\right|\cdot\frac{\left|\dfrac{P_{n+1}(x_1)}{P_{n+1}(x_0)}-1\right|}{\left|\dfrac{P_n(x_1)}{P_n(x_0)}-1\right|}.$$

Denote by $l(x_1, x_0)$ the greatest limit (the superior limit) of $R_n(x_1, x_0)$ as n becomes infinite.

Now if $l(x_1, x_0)$ depends explicitly upon x_1 and we determine x_1 so that $l(x_1, x_0)\leqq 1-\epsilon$, where ϵ is a positive quantity, we are assured that our series $\sum_n |\beta_{n+1}-\beta_n|$, and hence that our series $\mathrm{T}(x_1)$, converges. Moreover, if this inequality holds uniformly for x_1 in a given closed region containing no exceptional points and no limit point of exceptional points either in its interior or on its boundary, then the series $T(x_1)$ converges uniformly in this region. Moreover, if $l(x_1, x_0)$ can be written in the form $\theta(x_1)/\theta(x_0)$, then it is easy to show that a number λ exists such that $\theta(x) = \lambda$ is the boundary of the region of convergence in the sense that $T(x)$ converges for every non-exceptional point for which $\theta(x) < \lambda$ and diverges for every non-exceptional point for which $\theta(x) > \lambda$. Again if $l(x_1, x_0)$ can be written in the form $\theta(x_1) - \theta(x_0) + 1$, we may likewise readily derive a similar result.

Now if $l(x_1, x_0)$ is independent of x_1 and x_0 and is greater than or equal to unity we get no information concerning the convergence of $T(x_1)$. But if $l(x_1, x_0)$ is independent of x_1 and x_0 and has a value less than unity we conclude that $T(x_1)$ is convergent without further restriction on x_1. Series of sort therefore have the interesting property that if they converge for a single non-exceptional value of x they converge for every non-exceptional value.

In one of the cases in which the foregoing argument fails, namely, that in which $\lim_{n=\infty} R_n(x_1, x_0) = 1$, we may profitably proceed to a consideration of the greatest limit of the quantity

$$n\{R_n(x_1, x_0) - 1\}.$$

We denote this greatest limit by $l_1(x_1, x_0)$. If it is less than or equal to $-1 - \epsilon$ where ϵ is a positive constant, then the series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ converges, whence we conclude that $T(x_1)$ also converges.

Now if $l_1(x_1, x_0)$ depends on x_1 and we choose x_1 so that $l_1(x_1, x_0) \leqq -1 - \epsilon$ we have a situation similar to that just treated above and as before we can proceed readily to the determination of the character of the region of convergence, at least when $l_1(x_1, x_0)$ can be written as the quotient of a function of x_1, by a function of x_0, or $l_1(x_1, x_0) - 1$ as a difference of such functions.

Again, when $l_1(x_1, x_0)$ is independent of x_1 and x_0 we may treat the problem in the way indicated for the similar case above.

Out of other general criteria for the absolute convergence of series, as applied to the series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$, we may derive other related results. Those which we have stated have actually arisen frequently in special form in the investigation of the convergence of particular classes of series.*

It is desirable to examine certain special cases in which the foregoing greatest limits exist in such way as to give rise to an elegant theory.

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* One desiring to examine these special cases will find some of them treated and the others referred to in two papers of mine, namely, those in *Bull. Amer. Math. Soc.* (2) 8 (1917): 407-425 and AMER. JOURN. MATH., 36 (1914): 267-288, and in a paper by E. Cotton in *Bull. Soc. Math. Fr.*, 46 (1919): 69-84. The last paper is interesting for its general theorems, some of which are to be associated with the results of this section.

Let us suppose that

$$\lim_{n=\infty} \frac{P_n(x_1)}{P_n(x_0)}$$

exists, and let us denote its value by $m(x_1, x_0)$. Then if $m(x_1, x_0) \neq 1$ we have $\lim_{n=\infty} R_n(x_1, x_0) = | m(x_1, x_0) |$ so that $| m(x_1, x_0) |$ is to be identified with the $l(x_1, x_0)$ of the preceding discussion. An instance of this sort is afforded by a certain class of expansions in polynomials. Thus if we have

$$P_n(x) = a_{1n}(x-a) + a_{2n}(x-a)^2 + \cdots + a_{kn}(x-a)^k + \cdots a_{mn}(x-a)^m$$

where one of the coefficients a_{kn} dominates the others (in case there are any) in the sense that the quotient of any other by a_{kn} approaches zero as n becomes infinite, it is clear that $m(x_1, x_0) = (x_1 - a)^k / (x_0 - a)^k$. We see readily that the region of convergence is bounded by a circle about the point a as a center. For the special case of this in which $P_n(x) = x - a$ we have the usual ascending power series in $x - a$. By taking $P_n(x)$ a polynomial in $(x - a)^{-1}$ we obtain a like generalization of the usual descending power series in $x - a$.

This obviously may be further generalized by taking $P_n(x)$ in the form

$$P_n(x) = a_{1n}u_1(x) + \cdots + a_{kn}u_k(x) + \cdots + a_{mn}u_n(x)$$

where $u_1(x), \cdots, u_m(x)$ are given functions of x and the coefficient a_{kn} dominates the others in the same sense as before. The boundary of the region of convergence is now defined by an equation of the form $| u_k(x) | = \lambda$, and the region of convergence is on that side of this curve for which $| u_k(x) | < \lambda$. [It is clear that the finite series for $P_n(x)$ may be replaced by an infinite series if suitable hypotheses are made as to the character of its convergence.]

Let us now consider the case in which the foregoing limit value $m(x_1, x_0)$ is unity. Suppose that $P_n(x_1)/P_n(x_0)$ may be written in the form

$$\frac{P_n(x_1)}{P_n(x_0)} = 1 + \frac{m_1(x_1, x_0)}{n} + \frac{\xi_n(x_1, x_0)}{n^{1+\epsilon}},$$

where ϵ is a positive constant and $\xi_n(x_1, x_0)$ is bounded when n becomes infinite. Then we have

$$\lim_{n=\infty} n\{R(x_1, x_0) - 1\} = \lim_{n=\infty} n \left[\left| \frac{P_n(x_1)}{P_n(x_0)} \right| \cdot \left| \frac{\dfrac{m_1(x_1, x_0)}{n+1} + \dfrac{\xi_{n+1}(x_1, x_0)}{(n+1)^{1+\epsilon}}}{\dfrac{m_1(x_1, x_0)}{n} + \dfrac{\xi_{n+1}(x_1, x_0)}{n^{1+\epsilon}}} \right| - 1 \right]$$

$$= \lim_{n=\infty} n \left[\left\{ 1 + \frac{R\{m_1(x_1, x_0)\}}{n} \right\} \{1 - (1/n)\} - 1 \right]$$

$$= R\{m_1(x_1, x_0)\} - 1.$$

Hence $R\{m_1(x_1, x_0)\} - 1$ may be identified with the limit $l_1(x_1, x_0)$ of the preceding more general treatment; and the relevant convergence properties are therefore expressible in simple and elegant form.

It may be observed that it is essentially the Gauss criterion of convergence which lies at the bottom of the special result just obtained. Significant extensions of this result may be secured, if needed, by using a more delicate criterion than that of Gauss, say one of the infinite sequence of criteria due to Kummer (see *Encyclopédie des Sc. Math.*, I, p. 223).

The condition last indicated in detail is realized in the case of factorial series. Here we have $P_n(x) = 1/(x + n - 1) = \Gamma(x + n - 1)/\Gamma(x + n)$ so that

$$\frac{P_n(x_1)}{P_n(x_0)} = \frac{x_0 + n - 1}{x_1 + n - 1} = 1 + \frac{x_0 - x_1}{n} + \frac{\xi_n(x_1, x_0)}{n^2} :$$

Hence if the factorial series converges for a non-exceptional point x_0 it converges also for the non-exceptional point x_1 if $R(x_1) > R(x_0)$, as is well known.

It may be seen also that this condition is simply realized in a much more general class of cases. Let us put

$$P_n(x) = \frac{g(x + n)}{g(x + n - 1)}$$

where $g(x)$ is a given function possessing the asymptotic expansion

$$g(x) \sim x^{\mu + \sigma x} e^{\alpha + \beta x}(1 + (a_1/x) + \cdots), \qquad \sigma \neq 0,$$

valid in a sector V formed by two rays proceeding from 0 to ∞ and including between them the positive axis of reals. Then we have as to n an asymptotic relation of the form

$$P_n(x) \sim n^{\sigma} e^{\sigma - \beta}(1 + (\mu + \sigma x)/n + \cdots);$$

whence

$$\frac{P_n(x_1)}{P_n(x_0)} \sim 1 + \frac{\sigma(x_1 - x_0)}{n} + \cdots$$

From this we conclude that our series $T(x_1)$ in this case converges if $R(\sigma x_1) < R(\sigma x_0)$, and that the boundary of its region of convergence is a straight line $R(\sigma x) = \lambda$.

There is another range of cases, of which the generalized Dirichlet series affords an example, in which one may readily conclude to the convergence of

$\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ and hence of $T(x_1)$. Let us suppose that $\beta_{n+1} - \beta_n$ may be written in the form of an integral

$$\beta_{n+1} - \beta_n = \int_{C_n} u(t, x_1, x_0)\, dt$$

where C_n is a finite path of integration for each n, no two of these paths having a common arc. Let C be any path made up of all the paths C_l, C_{l+1} C_{l+2}, $\cdot \cdot \cdot$ (where l is a given integer) and any other paths which it is convenient to include. Then if the integral

$$\int_C \mid u(t, x_1, x_0) \mid dt$$

exists when x_1 is related in a specified way to x_0, it is clear that the series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$, and hence the series $T(x_1)$, converges under the same hypothesis as to the relation of x_1 and x_0.

If we take

$$P_1(x) = e^{-\lambda_1 x}, \quad P_n(x) = e^{-\lambda_n x + \lambda_{n-1} x} \text{ when } n > 1,$$

where $\lambda_1, \lambda_2, \lambda_3, \cdot \cdot \cdot$ is a monotone increasing sequence such that λ_n becomes infinite with n, our series $T(x)$ with $c_0 = 0$ is the Dirichlet series and the method indicated may be applied in the classic way illustrated already in § 2.

In the problem of representing functions with given properties or of finding functions with anticipated properties in given regions, it is sometimes desirable to have representations of them valid in certain preassigned regions of the plane. It is therefore of interest to ask under what simple conditions one will have a region of convergence of specified form: We add here a few remarks on this matter.

One of the simplest regions of convergence is a half plane bounded by a straight line. We have already observed certain cases in which the region of convergence is of this form, having the equation $R(\sigma x) = \lambda$. This suggests a more general case in which the same type of region of convergence arises. Let us suppose that the limit $m_1(x_1, x_0)$ of the preceding discussion is such that we have the relation

$$R\{m_1(x_1, x_0)\} = R\{\sigma(x_1 - x_0)\} \cdot \eta(x_1, x_0),$$

where $\eta(x_1, x_0)$ is positive whatever x_1 and x_0 are. Then it is easy to see that the region of convergence is bounded by a curve whose equation is of the form $R(\sigma x) = \lambda$.

From this special case it is clear that we may determine circumstances under which any one of a great variety of curves may be realized as the boundary of the region of convergence and that we may attach this discussion to any one of the four limit quantities $l(x_1, x_0)$, $l_1(x_1, x_0)$, $m(x_1, x_0)$, $m_1(x_1, x_0)$ employed above. We have already seen in particular how circular regions may be realized in a great variety of instances. For other circular regions and half-plane regions, see my papers already cited and the papers referred to in them.

§ 5. *Cases in which the Series Define Functions Having Formal Power Series Expansions.*

Owing to the great importance, in the theory of differential and difference equations and in other parts of analysis, of functions possessing formal power series expansions either convergent or divergent, it is desirable to know the circumstances under which our series $T(x)$ and $S(x)$ can be formally transformed into power series and these conversely into series $T(x)$ or $S(x)$.

Let us consider the case of series $T(x)$ where $P_n(x)$ has the descending formal power series expansion

$$P_n(x) = \frac{c_{1n}}{x} + \frac{c_{2n}}{x^2} + \frac{c_{3n}}{x^3} + \cdots, \quad n = 1, 2, 3, \cdots, \tag{7}$$

where C_{1n} is different from zero for every value of n. It is obvious that the series $T(x)$ is then transformable formally into a descending power series in x and that the coefficients may be reckoned out by means of readily solvable recurrence relations.

A simple case of such a series $T(x)$ is the factorial series in which $P_n(x) = 1/(x + n - 1)$. The generalization of factorial series introduced in § 4 by aid of the function $g(x)$ also belongs here when $\sigma = -1$, as one may show without difficulty. If we should take σ to be the negative reciprocal of an integer k we should have a generalization to the case in which the formal descending power series in x are replaced by formal descending power series in the kth root of x.

If we have two functions $T_1(x)$ and $T_2(x)$ defined by two series $T(x)$ each depending on the set of functions (7) and if the product $T_1(x)T_2(x)$ is expansible into a series $T(x)$ of the same form, then the coefficients of the expansion may be found in the following manner: Transform the series for $T_1(x)$ and $T_2(x)$ into formal descending power series in x, take the product of these latter and transform it formally into a series $T(x)$; this series will represent the product $T_1(x)T_2(x)$. This process will certainly be valid at least when $T_1(x)$ and $T_2(x)$ are asymptotic to their formal power series representations and the functions P are such that no function has two representations in the form of a series $T(x)$. An instance of this sort is afforded by the series defined in § 4 in terms of $g(x)$, provided that $\sigma = -1$, as one sees from the results in AMER. JOURN. MATH. 39 (1917): 385-403. It is not difficult to see that the case just treated is an instance of series $S(x)$ as well as of series $T(x)$.

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